The Relationship Between Loss, Conductivity, and Dielectric Constant

General Expressions

The question has been asked how loss, conductivity, and dielectric constant are interrelated. Answering this question requires a fairly extensive review of basic electromagnetics.

First, assume that one has a piece of arbitrary material. This material is made of atoms, molecules, or ions. Within this material exist electrons, either bound to individual atoms or free to move about. An electric field is applied across the object. The electrons will naturally want to move because of the electric field. The conduction electric current density (a measure of the flow of electrons) varies directly with the strength of the electric field. Thus

$$\overline{J_c} = \sigma_s \overline{E} \qquad \textbf{(1)}$$

where σ_s is a constant of proportionality, and it is called "conductivity." The conductivity provides a measure of how fast an electron can flow through a material. It is defined as

$$\sigma_s = -q\mu_e \qquad \textbf{(2)}$$

where q is the charge and μ_e is the electric mobility (not the permeability) of the medium.

Likewise, the electric flux density varies linearly with the application of the electric field so that

$$\overline{D} = \varepsilon \overline{E} \qquad \textbf{(3)}$$

Here, ε is the constant of proportionality, and it is called "permittivity."

The time-harmonic version of Maxwell's equations states that

$$\nabla \times \overline{H} = \overline{J} + j\omega \overline{D} \qquad \textbf{(4)}$$

$\overline{J}$ is the electric current density, and it has two parts. The first part is the impressed electric current density, $\overline{J_i}$ (that is, $\overline{J_i}$ is an excitation to the system by an outside source), and the second part is the aforementioned conduction electric current density, $\overline{J_c}$, caused by the application of an external electric field. Thus, we have

$$\nabla \times \overline{H} = \overline{J_i} + \overline{J_c} + j\omega \overline{D} \qquad \textbf{(5)}$$

$$\nabla \times \overline{H} = \overline{J_i} + \sigma_s \overline{E} + j\omega \overline{D} \qquad \textbf{(6)}$$

In most materials there exists at least one of three types of electric dipoles. Any kind of dipole exhibits a polarity; that is, one side of the dipole can be described as being negatively charged, and the other side can be described as being positively charged. The three types of dipoles are as follows.

1. Molecules arranged in such a way as to exhibit an imbalance of charge. For instance, water is bound in such a way that the two negative hydrogen atoms are on one side of the molecule, and a positive oxygen atom is on the other side. Hence, water has a net electric polarity.
2. Ions have inherently oppositely charged parts. For instance, table salt, NaCl, has a positive sodium atom (Na+) and a negative chlorine atom (Cl-).
3. Most atoms have a cloud of electrons surrounding the nucleus. Since the mass of an electron is much less than the mass of the nucleus, the application of an electric field causes the electrons to react and move much more quickly than the nucleus can react. The result is that the electron cloud shifts its position and is no longer centered about the nucleus. Hence, the atom ends with the positively charged nucleus on one side and the negatively charged electron cloud on the other side.

When an external electric field is applied, the dipoles align with the field. This action causes a term to be added to the electric flux density that has the same vector direction as the applied field. This relationship can be mathematically described as

$$\overline{D} = \varepsilon_0 \overline{E} + \varepsilon_0 \chi_e \overline{E} \qquad \textbf{(7)}$$

The term χ_e is known as the electric susceptibility and serves as a proportionality constant between the electric field and the portion of the electric flux density caused by the presence of the dielectric. One can rewrite the equation as

$$\overline{D} = \varepsilon_0 (1 + \chi_e) \overline{E} \qquad \textbf{(8)}$$

or

$$\overline{D} = \varepsilon_0 \varepsilon_r \overline{E} \qquad \textbf{(9)}$$

where ε_r is known as the relative permittivity of the medium.

ε_r is in general a complex quantity. To understand why, consider an alternating electric field applied to a dipole. When the field first strikes the dipole, the dipole rotates to align itself with the field. As time passes, the electric field reverses its direction, and the dipole must rotate again to remain aligned with the correct polarity. As it rotates, energy is lost through the generation of heat (friction) as well as the acceleration and deceleration of the rotational motion of the dipole. The degree to which the dipole is out of phase with the incident electric field and the losses that ensue determine how large the imaginary part of the permittivity is as a function of material and frequency. The larger the imaginary part, the more energy is being dissipated through motion, and the less energy is available to propagate past the dipole. Thus, the imaginary part of the relative permittivity directly relates to loss in the system.

To represent the real and imaginary parts of the absolute permittivity, the following convention is used.

$$\varepsilon_0 \varepsilon_r = \varepsilon' - j\varepsilon'' \tag{10}$$

Returning to Maxwell's equation **(6)**, we now have that

$$\nabla \times \overline{H} = \overline{J_i} + \sigma_s \overline{E} + j\omega(\varepsilon' - j\varepsilon'')\overline{E} \tag{11}$$

$$\nabla \times \overline{H} = \overline{J_i} + (\sigma_s + \omega\varepsilon'')\overline{E} + j\omega\varepsilon'\overline{E} \tag{12}$$

$$\nabla \times \overline{H} = \overline{J_i} + \sigma_e \overline{E} + j\omega\varepsilon'\overline{E} \tag{13}$$

In this last step, we have defined an effective conductivity,

$$\sigma_e = \sigma_s + \omega\varepsilon'' \tag{14}$$

The effective conductivity is the value that is usually specified in data sheets, although it might be labeled as merely "conductivity." The first term on the right-hand side of the above equation is the static conductivity, and we can define the last term to be conductivity due to an alternating field. Thus

$$\sigma_e = \sigma_s + \sigma_a \tag{15}$$

Again returning to Maxwell's equation **(13)**, we have now

$$\nabla \times \overline{H} = \overline{J_i} + j\omega\varepsilon'\left(1 - j\frac{\sigma_e}{\omega\varepsilon'}\right)\overline{E} \tag{16}$$

$$\nabla \times \overline{H} = \overline{J_i} + j\omega\varepsilon'(1 - j\tan\delta_e)\overline{E} \tag{17}$$

Here, we have defined the loss tangent, $\tan\delta_e$ as

$$\tan\delta_e = \frac{\sigma_e}{\omega\varepsilon'} \tag{18}$$

We can also expand Maxwell's equation **(16)** as

$$\nabla \times \overline{H} = \overline{J_i} + j\omega\varepsilon'\left(1 - j\frac{\sigma_s}{\omega\varepsilon'} - j\frac{\varepsilon''}{\varepsilon'}\right)\overline{E} \tag{19}$$

This last equation highlights the fact that two terms contribute to the loss tangent. The first term, $\frac{\sigma_s}{\omega\varepsilon'}$, describes loss due to collisions of electrons with other electrons and atoms. For instance, if the static conductivity is high (copper has $\sigma_s = 5.8x10^7 \; S/m$), then charges flow very easily without many collisions. At first glance it seems strange that a term that approaches infinity in the numerator describes a low loss structure, but it must be remembered that infinite conductivity implies zero electric field (and finite current density). That is, instead of viewing the current density as a function of the electric field,

$$\overline{J_c} = \sigma_s \overline{E} \qquad \textbf{(20)}$$

view the electric field as a function of the current density,

$$\overline{E} = \frac{\overline{J_c}}{\sigma_s} \qquad \textbf{(21)}$$

Now infinite conductivity makes sense. As one might expect, in conductors, this term of **(19)**, $\frac{\sigma_s}{\omega\varepsilon'}$, dominates the other term of **(19)**, $\frac{\varepsilon''}{\varepsilon'}$.

The $\frac{\varepsilon''}{\varepsilon'}$ term of **(19)** describes how much energy supplied by an external electric field is dissipated as motion and heat. In dielectrics, this term usually dominates the first term. In metals, the real part of the permittivity is usually equal to the permittivity of free space, and the imaginary part is usually zero. Semiconductors maintain a relative balance between the two terms.

Thus, when an effective conductivity is specified on a data sheet, it is useful to remember that it arises from two sources. For a metal, the effective conductivity is due almost entirely to the collisions of electrons, and the polarization dependent term is dropped. Maxwell's equation **(19)** reduces to

$$\nabla \times \overline{H} \approx \overline{J_i} + (j\omega\varepsilon_0 + \sigma_s)\overline{E} \qquad \textbf{(22)}$$

For a dielectric, the effective conductivity is due almost entirely to polarization loss (dipole motion), and the first term is dropped from the calculation. Maxwell's equation **(19)** becomes

$$\nabla \times \overline{H} \approx \overline{J_i} + j\omega\varepsilon'\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)\overline{E} \qquad \textbf{(23)}$$

We turn now to calculating the power absorbed and transmitted by a medium. We can begin with Maxwell's equations and find the following relationships. These equations are derived assuming that phasors represent the fields, and a dependence of $e^{j\omega t}$ is suppressed. Note that if these equations were derived using time derivatives, the results would be different. That is, one can not merely replace $\frac{\partial}{\partial t}$ by $j\omega$ in the result because of the non-linear nature of the equations (products of fields). For this reason, the Poynting vector, usually represented as $\overline{P} = \overline{E} \times \overline{H}$, here has the form $\overline{P} = \overline{E} \times \overline{H}^*$. If one uses the usual (non-phasor) derivation, then the results must be averaged over time to obtain the results here.

$$P_s = P_e + P_d + j2\omega\left(W_m + W_e\right) \quad \textbf{(24)}$$

$$P_s = -\frac{1}{2}\iiint_V \left(\overline{H}^* \cdot \overline{M}_i + \overline{E} \cdot \overline{J}_i^*\right) dv \quad \textbf{(25)}$$

$$P_e = \frac{1}{2}\oint\!\!\oint_S \left(\overline{E} \times \overline{H}^*\right) \cdot d\overline{s} \quad \textbf{(26)}$$

$$P_d = \frac{1}{2}\iiint_V \overline{J} \cdot \overline{E}^* \, dv = \frac{1}{2}\iiint_V \sigma_s \overline{E} \cdot \overline{E}^* \, dv = \frac{1}{2}\iiint_V \sigma_s \left|\overline{E}\right|^2 dv = \frac{1}{2}\iiint_V \frac{\left|\overline{J}\right|^2}{\sigma_s} dv \quad \textbf{(27)}$$

$$W_m = \frac{1}{4}\iiint_V \mu \left|\overline{H}\right|^2 dv \quad \textbf{(28)}$$

$$W_e = \frac{1}{4}\iiint_V \varepsilon \left|\overline{E}\right|^2 dv \quad \textbf{(29)}$$

In these equations, P_s represents the complex supplied power, P_e represents the complex exiting (transmitted) power, P_d represents the real dissipated power, and W_m and W_e represent stored energies. Usually these last two terms are strictly imaginary, but if μ or ε are complex, then either or both terms may be complex. The real parts of these terms can be extracted and added to the dissipated power. Here we ignore the possibility of complex permeability and rewrite the equation set as follows.

$$P_s = P_e + P_d + j2\omega\left(W_m + W_e\right) \quad \textbf{(30)}$$

$$P_s = -\frac{1}{2}\iiint_V \left(\overline{H}^* \cdot \overline{M}_i + \overline{E} \cdot \overline{J}_i^*\right) dv \quad \textbf{(31)}$$

$$P_e = \frac{1}{2} \oint\!\!\oint_S \left(\overline{E} \times \overline{H}^* \right) \cdot d\overline{s} \tag{32}$$

$$P_d = \frac{1}{2} \iiint_V \left(\sigma_s + \omega \varepsilon'' \right) \left| \overline{E} \right|^2 \, dv = \frac{1}{2} \iiint_V \sigma_e \left| \overline{E} \right|^2 \, dv \tag{33}$$

$$W_m = \frac{1}{4} \iiint_V \mu \left| \overline{H} \right|^2 \, dv \tag{34}$$

$$W_e = \frac{1}{4} \iiint_V \varepsilon' \left| \overline{E} \right|^2 \, dv \tag{35}$$

We are concerned here primarily with the dissipated power term, P_d. As mentioned before, it is puzzling to see that the dissipated power varies directly with the static conductivity. Rewriting P_d in terms of the current density helps the intuition.

$$P_d = \frac{1}{2} \iiint_V \left(\frac{\sigma_s + \omega \varepsilon''}{\sigma_s^2} \right) \left| \overline{J} \right|^2 \, dv \tag{36}$$

We now wish to compare the dissipated power with the exiting power and the stored powers. As an example, we choose a block of dielectric material that has a plane wave propagating inside. This wave originated from outside the block, so the impressed sources, $\overline{J_i}$ and $\overline{M_i}$, are zero. The plane wave enters at $z = 0$ and exits at $z = z_0$. Thus, P_e represents both the entering and the exiting power depending on the surface at which it is evaluated. P_d is the amount of power dissipated in the medium, and W_m and W_e are the stored energies.

Before we proceed, we note that non-zero but finite static conductivity implies that charges are present within the medium. This fact is true because non-zero finite static conductivity implies that charges take a finite time to travel through the medium, so that the time-averaged charge is non-zero. Zero static conductivity implies that free charges never enter the medium. Infinite static conductivity implies that charges progress instantaneously through the medium; thus, the time-averaged charge is again zero.

Often books talk about a “source-free lossy medium.” By “source-free” these books mean, in part, that the static conductivity is either zero or infinite. That is, the time-averaged charge is zero. By “lossy” these same books mean that there is polarization loss present in the material. That is, the permittivity has a non-zero imaginary part.

When considering a material with a finite non-zero conductivity, it would seem that discussion about a “source-free” medium could not apply. However, because the charges can be expressed in terms of the electric field ($\overline{J_c} = \sigma_s \overline{E}$), the problem can be treated by the same techniques used to treat source-free media. This fact is very fortunate, otherwise materials with non-zero static conductivities could only be treated numerically. As it is, we will now derive the closed-form solution to a plane wave traveling in a medium with finite non-zero static conductivity and with polarization loss.

The wave equation that we must solve is

$$\nabla^2 \overline{E} + k^2 \overline{E} = j\omega\mu\sigma_s \overline{E} \tag{37}$$

or

$$\nabla^2 \overline{E} + \left(k^2 - j\omega\mu\sigma_s\right)\overline{E} = 0 \tag{38}$$

Recalling that

$$k^2 = \omega^2 \mu\varepsilon = \omega^2 \mu\left(\varepsilon' - j\varepsilon''\right) \tag{39}$$

one obtains

$$\nabla^2 \overline{E} + \omega^2 \mu\varepsilon'\left(1 - j\left(\frac{\varepsilon''}{\varepsilon'} + \frac{\sigma_s}{\omega\varepsilon'}\right)\right)\overline{E} = 0 \tag{40}$$

or

$$\nabla^2 \overline{E} + \omega^2 \mu\varepsilon'\left(1 - j\frac{\sigma_e}{\omega\varepsilon'}\right)\overline{E} = 0 \tag{41}$$

or

$$\nabla^2 \overline{E} + \omega^2 \mu\varepsilon'\left(1 - j\tan\delta_e\right)\overline{E} = 0 \tag{42}$$

or

$$\nabla^2 \overline{E} + \gamma^2 \overline{E} = 0 \tag{43}$$

where

$$\gamma^2 = \omega^2 \mu\varepsilon'\left(1 - j\tan\delta_e\right) \tag{44}$$

A solution to the wave equation is

$$\overline{E} = E_0 e^{-j\gamma z}\hat{x} \tag{45}$$

γ can be found as follows.

$$\gamma = \sqrt{\omega^2 \mu \varepsilon' (1 - j \tan \delta_e)} \quad \textbf{(46)}$$

$$\gamma = \omega \sqrt{\mu \varepsilon'} (1 + \tan^2 \delta_e)^{1/4} \left(\cos \frac{\delta_e}{2} - j \sin \frac{\delta_e}{2} \right) \quad \textbf{(47)}$$

$$\gamma = \omega \sqrt{\mu \varepsilon'} \frac{1}{\sqrt{\cos \delta_e}} \left(\cos \frac{\delta_e}{2} - j \sin \frac{\delta_e}{2} \right) \quad \textbf{(48)}$$

$$\gamma = \omega \sqrt{\mu \varepsilon'} \left(\sqrt{\frac{1 + \cos \delta_e}{2 \cos \delta_e}} - j \sqrt{\frac{1 - \cos \delta_e}{2 \cos \delta_e}} \right) \quad \textbf{(49)}$$

The solution to the wave equation is then

$$\overline{E} = E_0 \, e^{-\omega \sqrt{\mu \varepsilon'} \sqrt{\frac{1 - \cos \delta_e}{2 \cos \delta_e}} z} e^{-j \omega \sqrt{\mu \varepsilon'} \sqrt{\frac{1 + \cos \delta_e}{2 \cos \delta_e}} z} \, \hat{x} \quad \textbf{(50)}$$

Letting

$$\alpha = \omega \sqrt{\mu \varepsilon'} \sqrt{\frac{1 - \cos \delta_e}{2 \cos \delta_e}} \quad \textbf{(51)}$$

and

$$\beta = \omega \sqrt{\mu \varepsilon'} \sqrt{\frac{1 + \cos \delta_e}{2 \cos \delta_e}} \quad \textbf{(52)}$$

then

$$\gamma = \beta - j \alpha \quad \textbf{(53)}$$

$$\overline{E} = E_0 \, e^{-\alpha z} e^{-j \beta z} \, \hat{x} \quad \textbf{(54)}$$

Notice that if $\sigma_s = 0$ and $\varepsilon'' = 0$ then $\delta_e = 0$ so that

$$\alpha = 0 \quad \textbf{(55)}$$

$$\beta = k \quad \textbf{(56)}$$

and

$$\overline{E} = E_0\, e^{-jkz}\, \hat{x} \qquad \textbf{(57)}$$

which is the usual expression for a plane wave in lossless space. The magnetic field can be found from

$$\nabla \times \overline{E} = -j\omega\mu\overline{H} \qquad \textbf{(58)}$$

or

$$-j\omega\mu\overline{H} = -j\gamma E_0\, e^{-j\gamma z}\, \hat{y} \qquad \textbf{(59)}$$

or

$$\overline{H} = \frac{\gamma}{\omega\mu} E_0\, e^{-j\gamma z}\, \hat{y} \qquad \textbf{(60)}$$

This expression, **(60),** is only equal to

$$\overline{H} = \frac{\sqrt{\varepsilon}}{\sqrt{\mu}} E_0\, e^{-j\gamma z}\, \hat{y} = \frac{1}{\eta} E_0\, e^{-j\gamma z}\, \hat{y} \qquad \textbf{(61)}$$

when $\sigma_s = 0$, a condition that is not generally true. We now summarize the equations of the fields.

$$\overline{E} = E_0\, e^{-\alpha z} e^{-j\beta z}\, \hat{x} \qquad \textbf{(62)}$$

$$\overline{H} = \frac{\beta - j\alpha}{\omega\mu} E_0\, e^{-\alpha z}\, e^{-j\beta z}\, \hat{y} \qquad \textbf{(63)}$$

The next step is to compute the desired powers.

$$P_e = \frac{1}{2} \oiint_S \left(\overline{E} \times \overline{H}^*\right) \cdot d\overline{s} = \frac{1}{2} \oiint_S E_0^2\, \frac{\beta + j\alpha}{\omega\mu}\, e^{-2\alpha z}\, \hat{z} \cdot d\overline{s} \qquad \textbf{(64)}$$

$$P_d = \frac{1}{2} \iiint_V \sigma_e \left|\overline{E}\right|^2 dv = \frac{1}{2} \iiint_V \sigma_e\, E_0^2\, e^{-2\alpha z}\, dv \qquad \textbf{(65)}$$

$$W_m = \frac{1}{4} \iiint_V \mu \left|\overline{H}\right|^2 dv = \frac{1}{4} \iiint_V \frac{\varepsilon'}{\cos\delta_e} E_0^2\, e^{-2\alpha z}\, dv \qquad \textbf{(66)}$$

$$W_e = \frac{1}{4}\iiint_V \varepsilon' \left|\overline{E}\right|^2 dv = \frac{1}{4}\iiint_V \varepsilon' E_0^2\, e^{-2\alpha z}\, dv \qquad \textbf{(67)}$$

For a block of material having length, z_0, and cross-sectional area, A, the integrals evaluate as follows.

The power entering the material at $z = 0$:

$$P_{in} = -P_e\big|_{z=0} = -\frac{1}{2}\oint\!\!\oint_S E_0^2\, \frac{\beta + j\alpha}{\omega\,\mu} e^{-2\alpha z}\, \hat{z}\cdot d\overline{s} = \frac{A}{2} E_0^2\, \frac{\beta + j\alpha}{\omega\,\mu} \qquad \textbf{(68)}$$

The power exiting the material at $z = z_0$:

$$P_{out} = P_e\big|_{z=z_0} = \frac{1}{2}\oint\!\!\oint_S E_0^2\, \frac{\beta + j\alpha}{\omega\,\mu} e^{-2\alpha z}\, \hat{z}\cdot d\overline{s} = \frac{A}{2} E_0^2\, \frac{\beta + j\alpha}{\omega\,\mu} e^{-2\alpha z_0} \qquad \textbf{(69)}$$

The power dissipated inside the material as heat:

$$P_d = \frac{1}{2}\iiint_V \sigma_e\, E_0^2\, e^{-2\alpha z}\, dv = \frac{A}{4\alpha}\, \sigma_e\, E_0^2 \left(1 - e^{-2\alpha z_0}\right) \qquad \textbf{(70)}$$

The power stored in magnetic fields:

$$W_m = \frac{1}{4}\iiint_V \frac{\varepsilon'}{\cos\delta_e} E_0^2\, e^{-2\alpha z}\, dv = \frac{A}{8\alpha}\frac{\varepsilon'}{\cos\delta_e} E_0^2 \left(1 - e^{-2\alpha z_0}\right) \qquad \textbf{(71)}$$

The power stored in electric fields:

$$W_e = \frac{1}{4}\iiint_V \varepsilon'\, E_0^2\, e^{-2\alpha z}\, dv = \frac{A}{8\alpha} \varepsilon' E_0^2 \left(1 - e^{-2\alpha z_0}\right) \qquad \textbf{(72)}$$

Note that P_{in} and P_{out} appear on the same side of equation **(30)**. In **(68)** the sign of P_{in} is reversed when it is defined to make the sum of the output, dissipated, and stored powers equal to the input power.

Taking the ratio of P_d to the real part of $P_{in} - P_{out}$, one obtains

$$\frac{P_d}{\mathrm{Re}\{P_{in} - P_{out}\}} = \frac{\sigma_e}{2\alpha}\frac{\omega\,\mu}{\beta} = \frac{\omega\,\mu\,\sigma_e}{2\omega^2\,\mu\,\varepsilon' \dfrac{\sqrt{1-\cos^2\delta_e}}{2\cos\delta_e}} = \frac{\sigma_e}{\omega\,\varepsilon' \tan\delta_e} = 1 \qquad \textbf{(73)}$$

Thus, the dissipated (real) power is equal to the difference between the input and output real powers. Likewise, one can show that

$$\frac{2\,\omega\left(W_m + W_e\right)}{\mathrm{Im}\left\{P_{in} - P_{out}\right\}} = 1 \qquad \textbf{(74)}$$

to prove that the stored (imaginary) power equals the difference in the input and output reactive powers. The ratio of the dissipated power to real input power is found as

$$\frac{P_d}{\mathrm{Re}\left\{P_{in}\right\}} = \frac{\dfrac{A}{4\alpha}\,\sigma_e\,E_0^2\left(1 - e^{-2\alpha z_0}\right)}{\dfrac{A}{2}\,E_0^2\,\dfrac{\beta}{\omega\,\mu}} \qquad \textbf{(75)}$$

$$\frac{P_d}{\mathrm{Re}\left\{P_{in}\right\}} = \frac{\omega\,\mu\,\sigma_e}{2\,\alpha\,\beta}\left(1 - e^{-2\,\alpha\,z_0}\right) = 1 - e^{-2\,\alpha\,z_0} \qquad \textbf{(76)}$$

If $\sigma_s = 0$ then the more general equations **(18)**, **(51)**, **(52)**, and **(62)** through **(67)** specialize as

$$\tan\delta_e = \frac{\varepsilon''}{\varepsilon'} + \frac{\sigma_s}{\omega\,\varepsilon'} = \frac{\varepsilon''}{\varepsilon'} \qquad \textbf{(77)}$$

$$\alpha = |k|\sin\frac{\delta_e}{2} \qquad \textbf{(78)}$$

$$\beta = |k|\cos\frac{\delta_e}{2} \qquad \textbf{(79)}$$

$$\overline{E} = E_0\,e^{-\alpha z}e^{-j\,\beta\,z}\,\hat{x} \qquad \textbf{(80)}$$

$$\overline{H} = \frac{1}{|\eta|}E_0\,e^{-\alpha z}e^{-j\,(\beta\,z+\theta)}\,\hat{y} \qquad \textbf{(81)}$$

where $$\eta = \sqrt{\frac{\mu}{\varepsilon}} = |\eta|\,e^{j\theta} \qquad \textbf{(82)}$$

$$P_e = \frac{1}{2}\oint\!\!\oint_S \left(\overline{E}\times\overline{H}^*\right)\cdot d\overline{s} = \frac{1}{2}\oint\!\!\oint_S \frac{E_0^2}{|\eta|}\,e^{-2\alpha z}\,e^{j\theta}\,\hat{z}\cdot d\overline{s} \qquad \textbf{(83)}$$

$$P_d = \frac{1}{2}\iiint_V \sigma_e \left|\overline{E}\right|^2 dv = \frac{1}{2}\iiint_V \omega\varepsilon'' E_0^2 e^{-2\alpha z} dv \qquad \textbf{(84)}$$

$$W_m = \frac{1}{4}\iiint_V \mu\left|\overline{H}\right|^2 dv = \frac{1}{4}\iiint_V |\varepsilon| E_0^2 e^{-2\alpha z} dv \qquad \textbf{(85)}$$

$$W_e = \frac{1}{4}\iiint_V \varepsilon' \left|\overline{E}\right|^2 dv = \frac{1}{4}\iiint_V \varepsilon' E_0^2 e^{-2\alpha z} dv \qquad \textbf{(86)}$$

As $\sigma_s \to \infty$, α and β become infinite, and the fields go to zero. The assumptions that $\sigma_s \neq 0$ and $\varepsilon'' = 0$ do not yield any reduction of the equations.

Primary Source: Balanis, Advanced Engineering Electromagnetics.